Models of Set Theory II - Winter 2015/2016

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Let $[\omega]^\omega = \{x \subseteq \omega \mid |x| = \aleph_0\}$ and $[\omega]^{<\omega} = \{s \subseteq \omega \mid |s| < \aleph_0\}$.

For $x, y \subseteq \omega$ we say that x is *almost contained* in y, denoted $x \subseteq^* y$, if $x \setminus y$ is finite. A *pseudo-intersection* of a family $\mathcal{F} \subseteq [\omega]^\omega$ is an element $x \in [\omega]^\omega$ such that for every $y \in \mathcal{F}$, $x \subseteq^* y$. Furthermore, we say that $\mathcal{F} \subseteq [\omega]^\omega$ has the *strong finite intersection property (sfip)*, if every finite subfamily of $\mathcal{F}$ has infinite intersection.

The *pseudo-intersection number* $\mathfrak{p}$ is defined as the least cardinality of a family $\mathcal{F} \subseteq [\omega]^\omega$ which has the sfip but does not have a pseudo-intersection.

Problem 23 (4 points)**.** Let $\mathcal{F}, \mathcal{G} \subseteq [\omega]^\omega$ be nonempty families of size $< \mathfrak{p}$ such that for all $y \in \mathcal{G}$, $\{x \cap y \mid x \in \mathcal{F}\}$ has the sfip.

(a) Let $\mathcal{F}^* = \{\bar{x} \mid x \in \mathcal{F}\} \cup \{\tilde{y} \mid y \in \mathcal{G}\} \cup \{z_n \mid n \in \omega\}$, where for $x \in \mathcal{F}, y \in \mathcal{G}$ and $n \in \omega$, $\bar{x} = \{s \in [\omega]^{<\omega} \mid s \subseteq x\}, \tilde{y} = \{s \in [\omega]^{<\omega} \mid s \cap y \neq \emptyset\}$ and $z_n = \{s \in [\omega]^{<\omega} \mid \min s > n\}$. Show that $\mathcal{F}^*$ has the sfip.

(b) Show that $\mathcal{F}$ has a pseudo-intersection x such that for each $y \in \mathcal{G}$, $x \cap y$ is infinite.

Problem 24 (6 points)**.** Let $\{I_n \mid n \in \omega\}$ be an enumeration of all open intervals in $\mathbb{R}$ with rational endpoints. Prove the following statements:

(a) Suppose that $\{D_\alpha \mid \alpha < \kappa\}$ is a set of dense open subsets of $\mathbb{R}$. Let $x_\alpha = \{n \in \omega \mid I_n \subseteq D_\alpha\}$ for $\alpha < \kappa$ and $y_k = \{n \in \omega \mid I_n \subseteq I_k\}$ for $k \in \omega$. Show that for each $k \in \omega$, $\{x_\alpha \cap y_k \mid \alpha < \kappa\}$ has the sfip.

(b) Show that $\aleph_1 \leq \mathfrak{p} \leq \operatorname{add}(\mathcal{M})$.

Problem 25 (6 points)**.** Let $\mathcal{F} \subseteq [\omega]^\omega$ be a family which satisfies the sfip and consider the forcing notion $\mathbb{P}_\mathcal{F}$ whose conditions are pairs $p = \langle s_p, E_p \rangle$ such that s_p is a finite subset of ω and E_p is a finite subset of $\mathcal{F}$, ordered by

$$p \leq q \iff s_p \supseteq s_q \wedge E_p \supseteq E_q \wedge s_p \setminus s_q \subseteq \bigcap E_q.$$

Prove the following statements:

(a) If G is M-generic for $\mathbb{P}_\mathcal{F}$ then in $M[G]$, $\mathcal{F}$ has a pseudo-intersection.

(b) MA implies that $\mathfrak{p} = 2^{\aleph_0}$.

Problem 26 (4 points)**.** Let $M \models \mathsf{ZFC} + \mathsf{CH} + 2^{\aleph_1} = \aleph_2$. Show that there is a finite support iteration of length ω_2 of forcing notions of the form $\mathbb{P}_{\mathcal{F}}$ as in Problem 25 such that $M[G] \models \mathfrak{p} = 2^{\aleph_0} = \aleph_2$.

Hint: Let $h : \omega_2 \times \omega_2 \to \omega_2$ denote Gödel pairing. Define $\mathbb{P}_\gamma$-names $\dot{\mathbb{Q}}_\gamma$ for a forcing notion and $\dot{F}_\gamma$ for such that $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash^M_{\mathbb{P}_\alpha} \dot{F}_\alpha :$ "$\check{\omega}_2 \to \mathcal{P}([\omega]^\omega)$ is a bijection". Suppose that $\dot{\mathbb{Q}}_\alpha, \dot{F}_\alpha$ are given for $\alpha < \gamma$. If $\gamma = h(\alpha, \beta)$ define $\dot{\mathbb{Q}}_\gamma$ using $\dot{F}_\alpha(\check{\beta})$.

Please hand in your solutions on Monday, 14.12.2015 before the lecture.